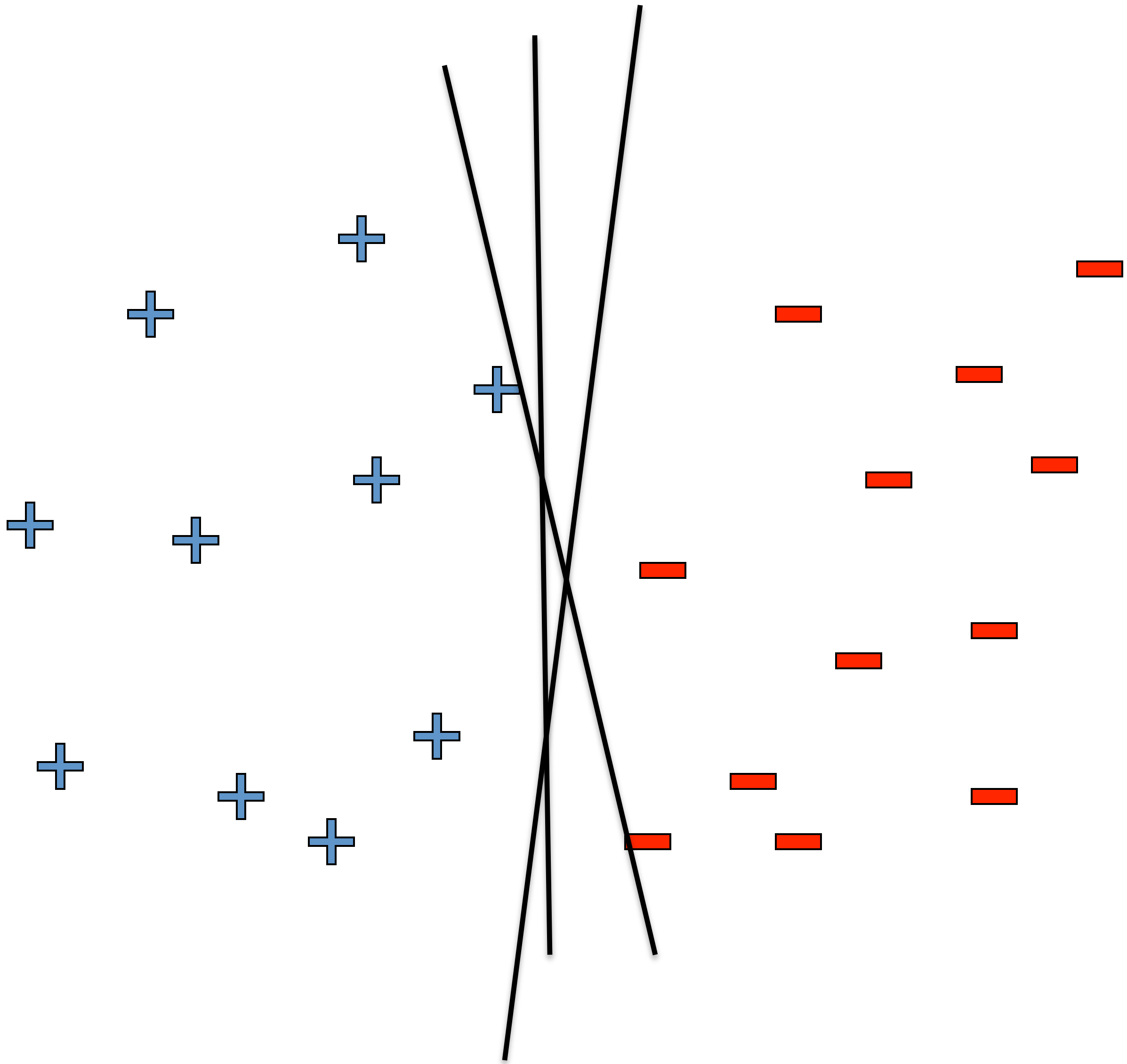
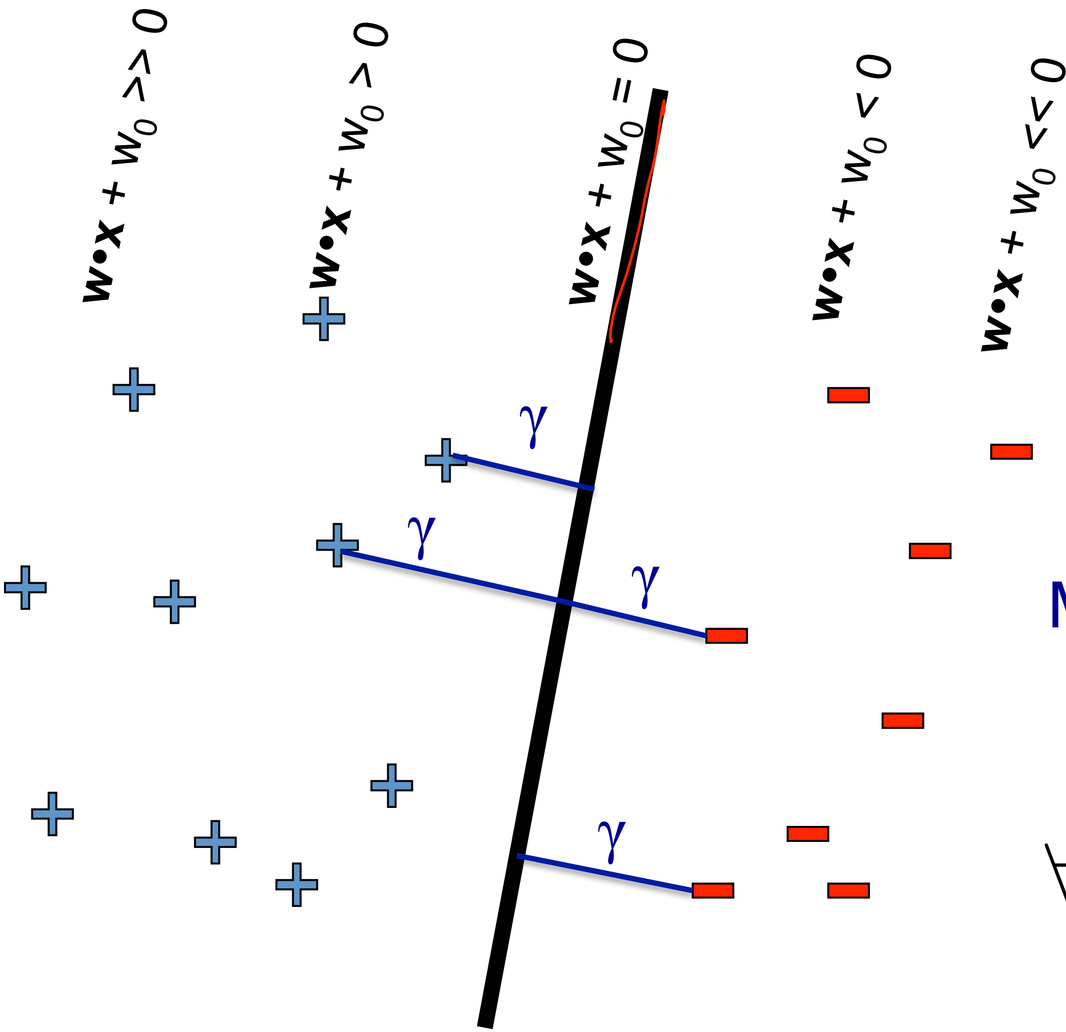
Linear classifiers – Which line is better?



Pick the one with the largest margin!



$\mathbf{w} \bullet \mathbf{x} = \sum_i w_i x_i$

Margin for point j:

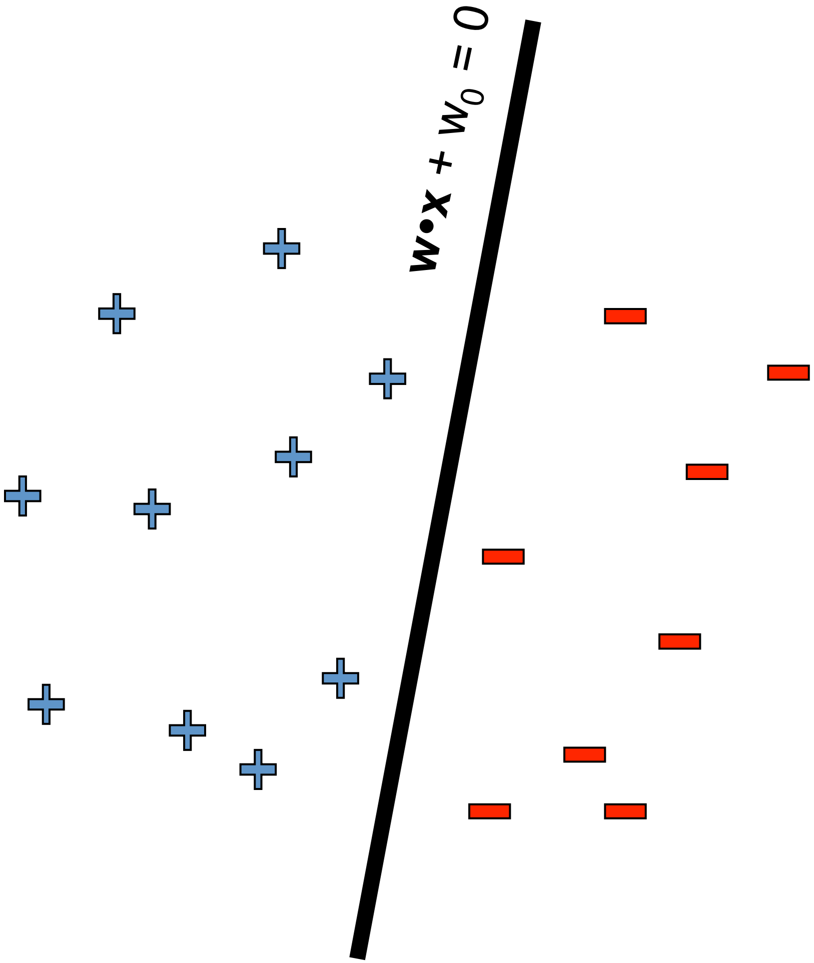
$$\gamma^j = y^j (w \cdot x^j + w_0)$$

Max Margin:

$$\max_{\gamma, w, w_0} \gamma$$

$$\forall j. y^j (w \cdot x^j + w_0) > \gamma$$

How many possible solutions?



$$\max_{\gamma, w, w_0} \gamma$$

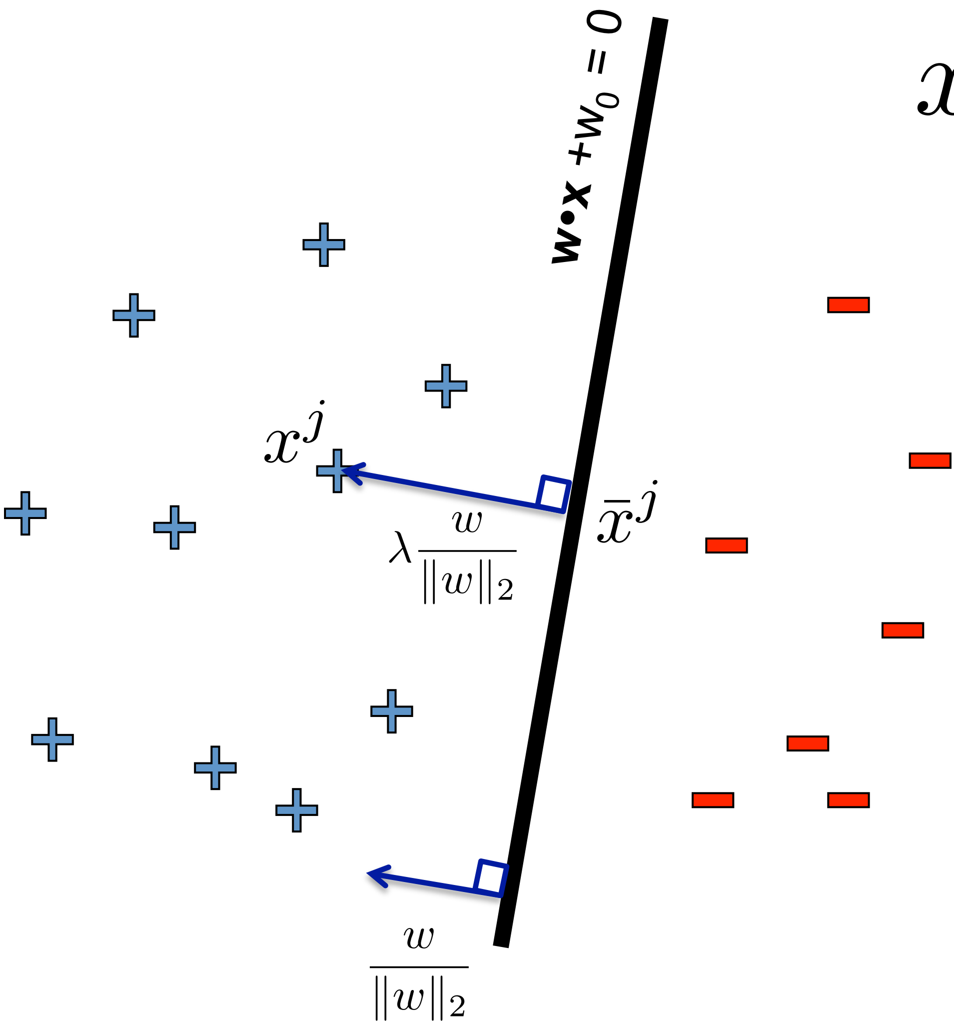
$$\forall j. y^j (w \cdot x^j + w_0) > \gamma$$

Any other ways of writing the same dividing line?

- **w.x** + b = 0
- 2**w.x** + 2b = 0
- 1000**w.x** + 1000b = 0
- ….
- Any constant scaling has the same intersection with z=0 plane, so same dividing line!

Do we really want to max $_{\gamma,w,w0}$?

Review: Normal to a plane



$$x^j = \bar{x}^j + \lambda \frac{w}{\|w\|_2}$$

Key Terms

$\bar{x}^j$ -- projection of x^j onto w

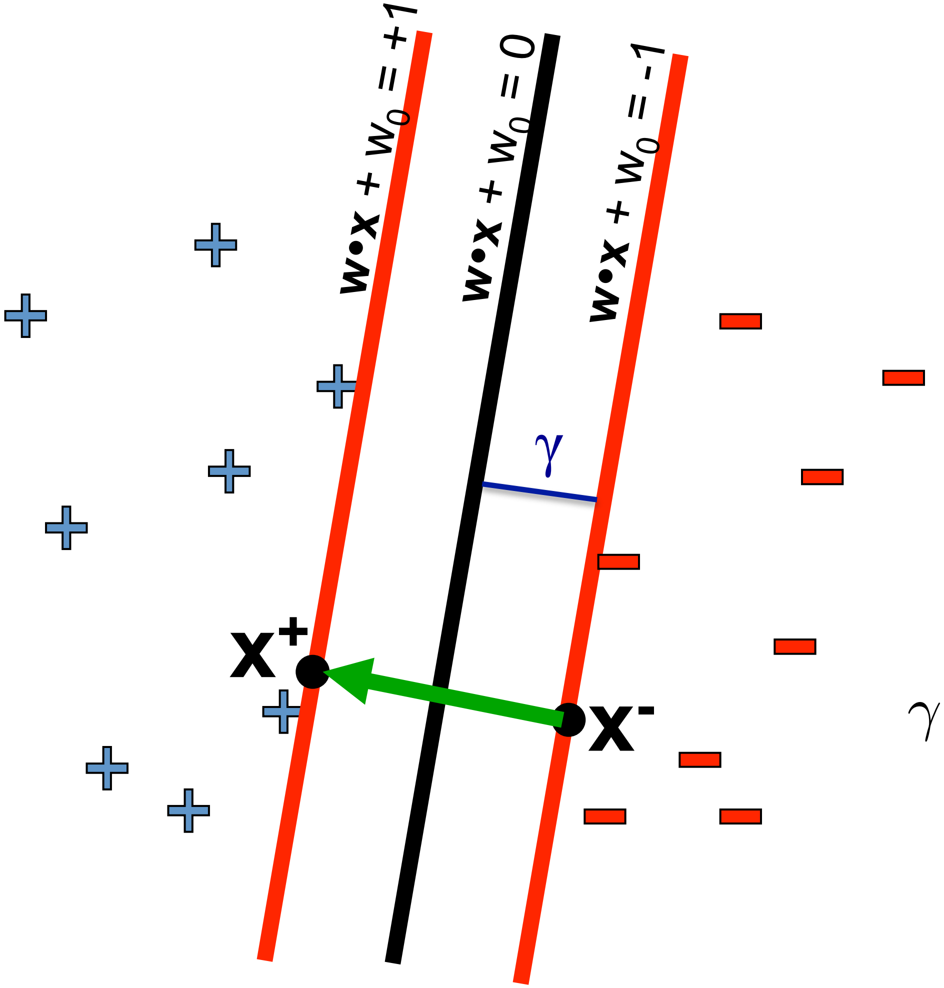
$\frac{w}{\|w\|_2}$ -- unit vector normal to w

$$\|w\|_2 = \sqrt{\sum_i w_i^2}$$

Idea: *constrained* margin

$$x^j = \bar{x}^j + \lambda \frac{w}{\|w\|_2}$$

$$\|w\|_2 = \sqrt{\sum_i w_i^2}$$



Assume: x^+ on positive line (y=1 intercept), x^- on negative (y=-1)

$$x^+ = x^- + 2\gamma \frac{w}{||w||}$$

$$w \cdot x^+ + w_0 = 1$$

$$w \cdot (x^- + 2\gamma \frac{w}{\|w\|_2}) + w_0 = 1$$

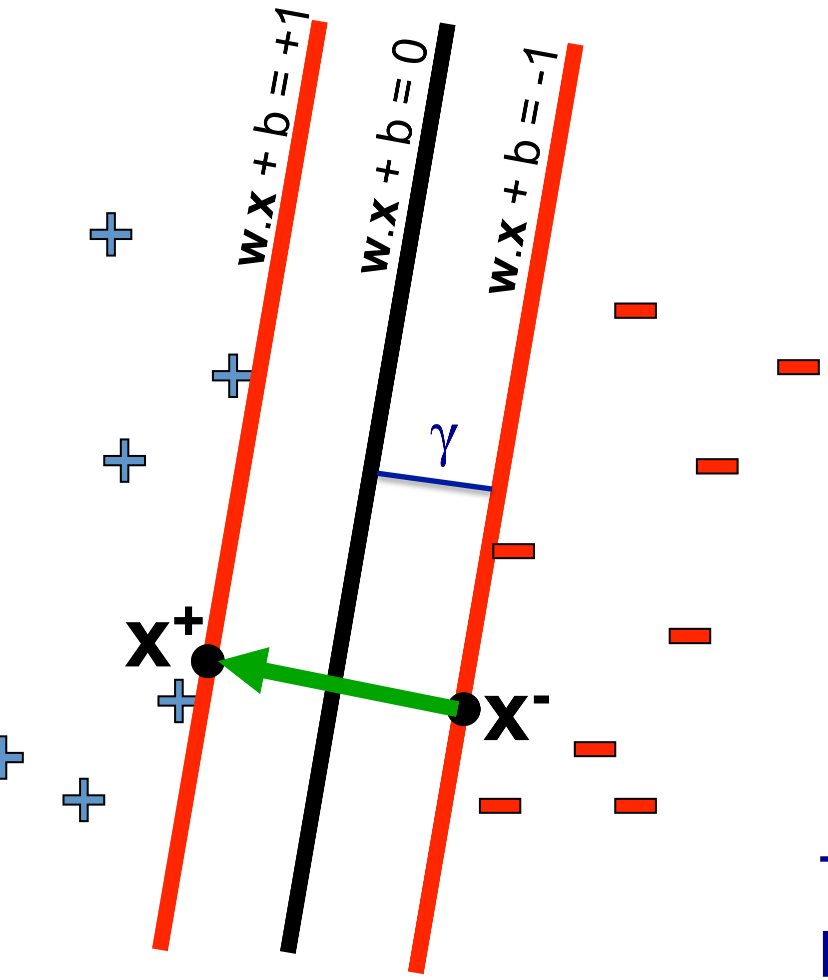
$$w \cdot x^- + w_0 + 2\gamma \frac{w \cdot w}{\|w\|_2} = 1$$

$$\gamma \frac{w \cdot w}{\|w\|_2} = 1 \qquad w \cdot w = \sum_i w_i^2 = \|w\|_2^2$$

$$\gamma = \frac{\|w\|_2}{w \cdot w} = \frac{1}{\|w\|_2}$$

Final result: can maximize constrained margin by minimizing $||w||_2$!!!

Max margin using canonical hyperplanes



$$\max_{\gamma, w, w_0} \gamma$$

$$\forall j. y^j (w \cdot x^j + w_0) > \gamma$$

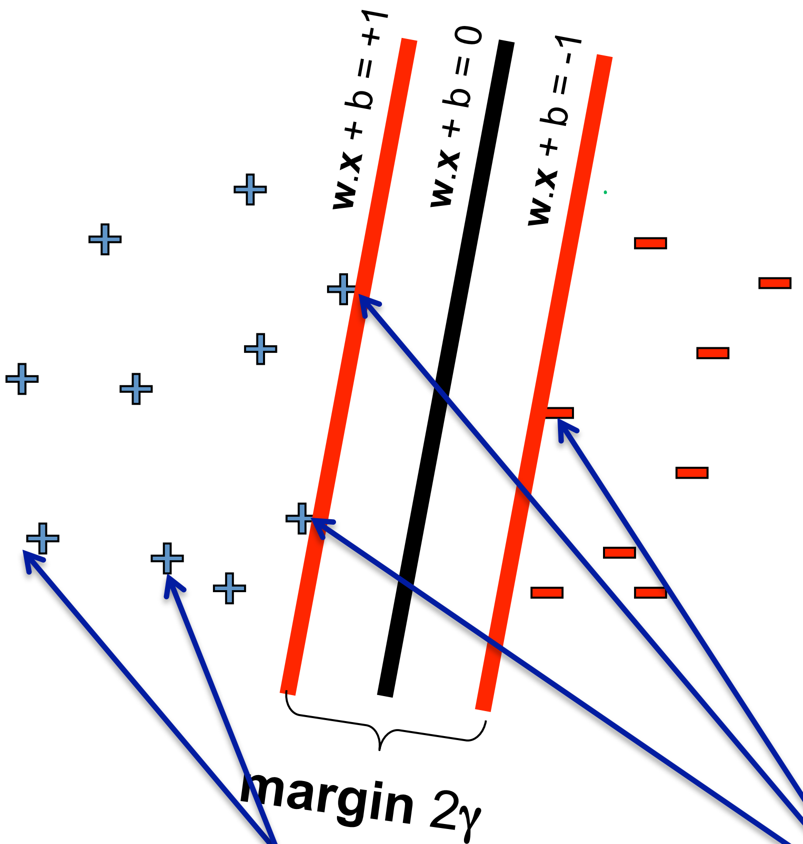
$$\gamma = \frac{1}{\|w\|_2}$$

$$\min_{w, w_0} \frac{1}{2} \|w\|_2^2$$

$$\forall j. y^j (w \cdot x^j + w_0) \geq 1$$

The assumption of canonical hyperplanes (at +1 and -1) changes the objective and the constraints!

Support vector machines (SVMs)



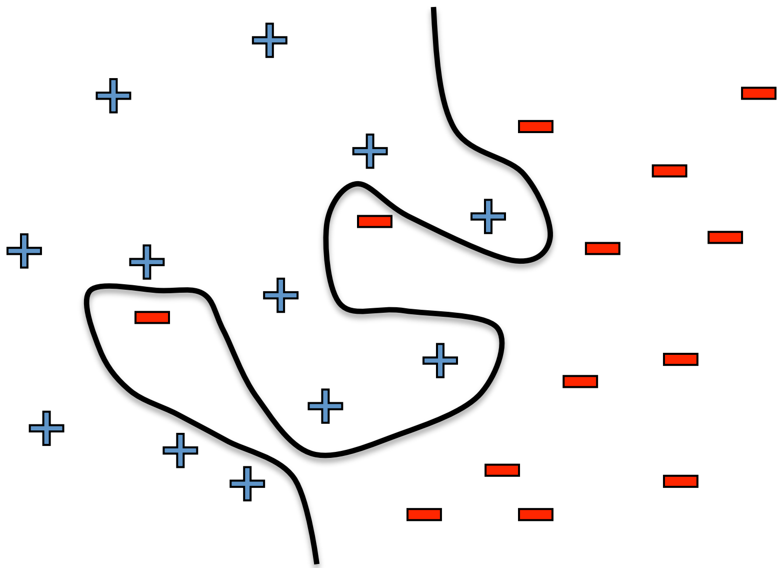
$$\min_{w,w_0} \frac{1}{2}\|w\|_2^2$$
$$\forall j . y^j (w \cdot x^j + w_0) \geq 1$$

- Solve efficiently by quadratic programming (QP)
  - Well-studied solution algorithms
  - Not simple gradient ascent, but close
- Decision boundary defined by support vectors

Non-support Vectors:
- everything else
- moving them will not change w

Support Vectors:
- data points on the canonical lines

What if the data is not linearly separable?



Add More Features!!!

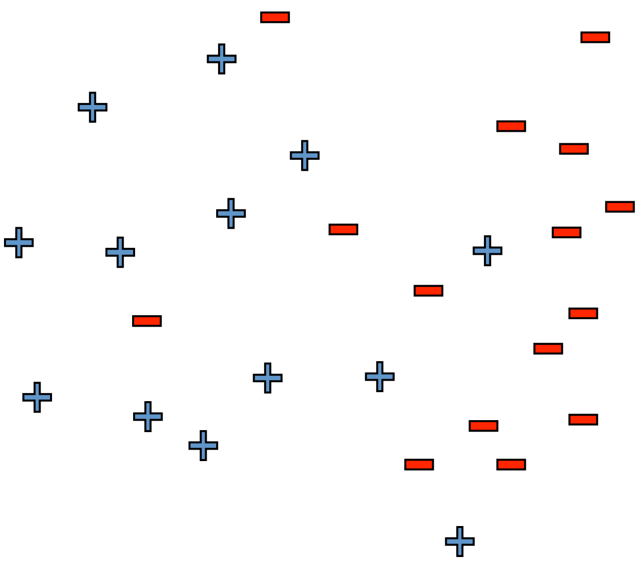
$$\phi(x) = \begin{pmatrix} x_1 \\ \dots \\ x_n \\ x_1 x_2 \\ x_1 x_3 \\ \dots \\ e^{x_1} \\ \dots \end{pmatrix}$$

Can use Kernels…
What about overfitting?

What if the data is still not linearly separable?

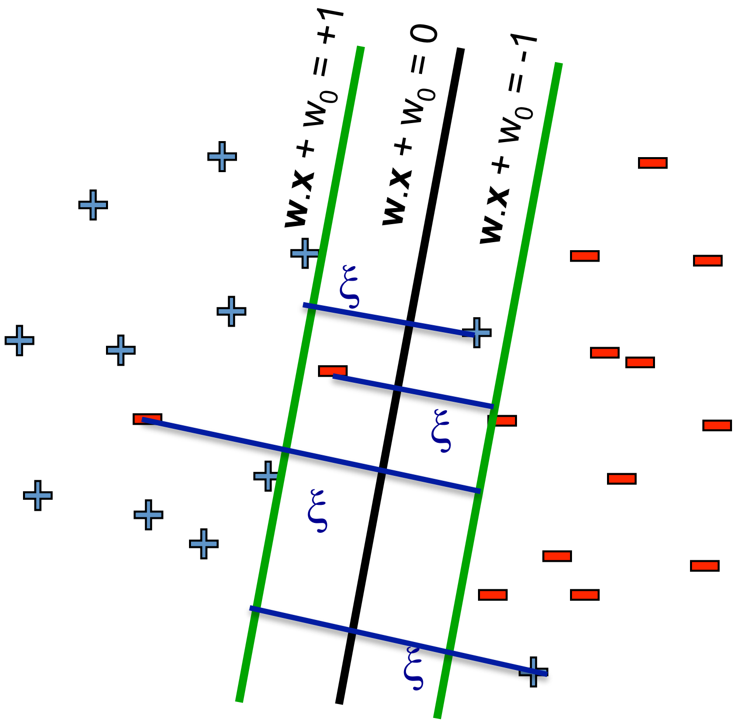
$$\min_{w,w_0} \frac{1}{2}\|w\|_2^2 \text{ + C \#(mistakes)}$$

$$\forall j. y^j (w \cdot x^j + w_0) \geq 1$$



- First Idea: Jointly minimize $\|w\|_2^2$ and number of training mistakes
  - How to tradeoff two criteria?
  - Pick C on development / cross validation
- Tradeoff #(mistakes) and $\|w\|_2^2$
  - 0/1 loss
  - Not QP anymore
  - Also doesn't distinguish near misses and really bad mistakes
  - NP hard to find optimal solution!!!

Slack variables – Hinge loss



$$\min_{w,w_0} \frac{1}{2}\|w\|_2^2 + C \Sigma_j \xi^j$$

$$\forall j . y^j(w \cdot x^j + w_0) \geq 1 - \xi^j \quad , \xi^j \geq 0$$

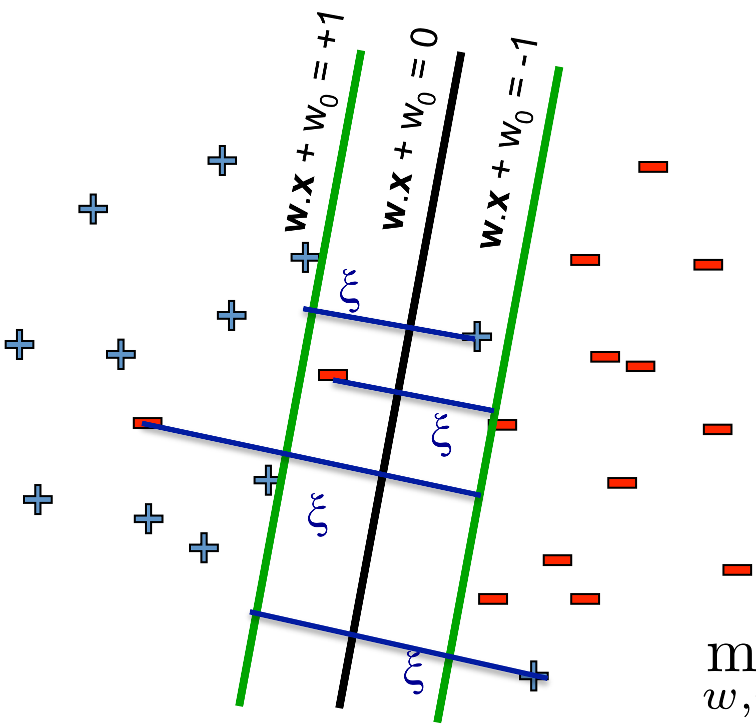
Slack Penalty $C > 0$:

- $C=\infty$ → have to separate the data!
- $C=0$ → ignore data entirely!
- Select on dev. set, etc.

For each data point:

- If margin ≥ 1, don't care
- If margin < 1, pay linear penalty

Slack variables – Hinge loss



$$\min_{w,w_0} \frac{1}{2}\|w\|_2^2 + C\,\Sigma_j\, \xi^j$$

$$\forall j.\, y^j(w \cdot x^j + w_0) \geq 1 - \xi^j \quad , \xi^j \geq 0$$

$[x]_+ = \max(x,0)$

$$\min_{w,w_0} \frac{1}{2}\|w\|_2^2 + C\sum_{j=1}^{N}[1 - y^j(w \cdot x^j + w_0)]_+$$

Regularization — Hinge Loss

Solving SVMs:
- Differentiate and set equal to zero!
- No closed form solution, but quadratic program (top) is concave
- Hinge loss is not differentiable, gradient ascent a little trickier…

Logistic Regression as Minimizing Loss

Logistic regression assumes:

$$f(x) = w_0 + \sum_i w_i x_i$$

$$P(Y=1|X=x) = \frac{\exp(f(x))}{1+\exp(f(x))}$$

And tries to maximize data likelihood, for Y={-1,+1}:

$$P(y^i|x^i) = \frac{1}{1+\exp(-y^i f(x^i))}$$

$$\ln P(\mathcal{D}_Y \mid \mathcal{D}_{\mathbf{X}}, \mathbf{w}) = \sum_{j=1}^{N} \ln P(y^j \mid \mathbf{x}^j, \mathbf{w})$$

$$= -\sum_{i=1}^{N} \ln(1+\exp(-y^i f(x^i)))$$

Equivalent to minimizing *log loss*:

$$\sum_{i=1}^{N} \ln(1+\exp(-y^i f(x^i)))$$

SVMs vs Regularized Logistic Regression

$$f(x) = w_0 + \sum_i w_i x_i$$

SVM Objective:

$$\arg\min_{\mathbf{w}, w_0} \frac{1}{2}\|\mathbf{w}\|_2^2 + C \sum_{j=1}^{N} [1 - y^j f(x^j)]_+$$

$[x]_+ = \max(x,0)$

Logistic regression objective:

$$\arg\min_{\mathbf{w}, w_o} \lambda\|\mathbf{w}\|_2^2 + \sum_{j=1}^{N} \ln(1 + \exp(-y^j f(x^j)))$$

Tradeoff: same l_2 regularization term, but different error term

Graphing Loss vs Margin

Logistic regression:

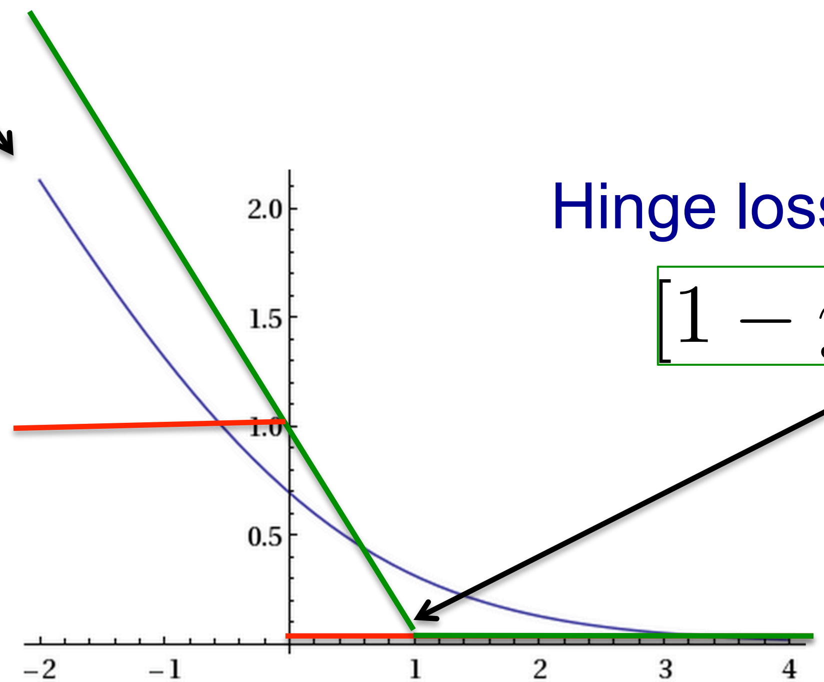
$$\ln(1 + \exp(-y^j f(x^j)))$$

Hinge loss:

$$[1 - y^j f(x^j)]_+$$

0-1 Loss:

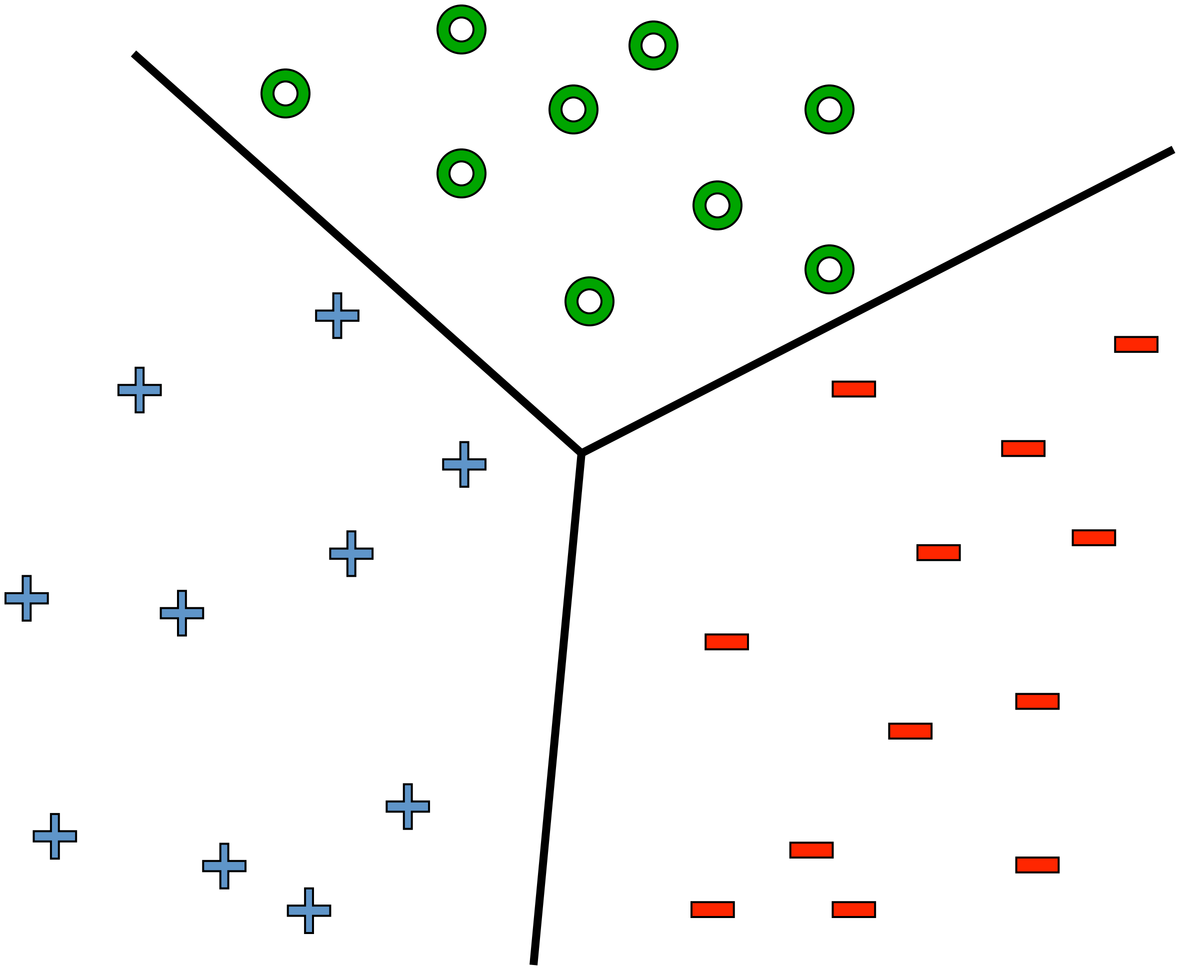
$$\delta(f(x^j) \neq y^j)$$



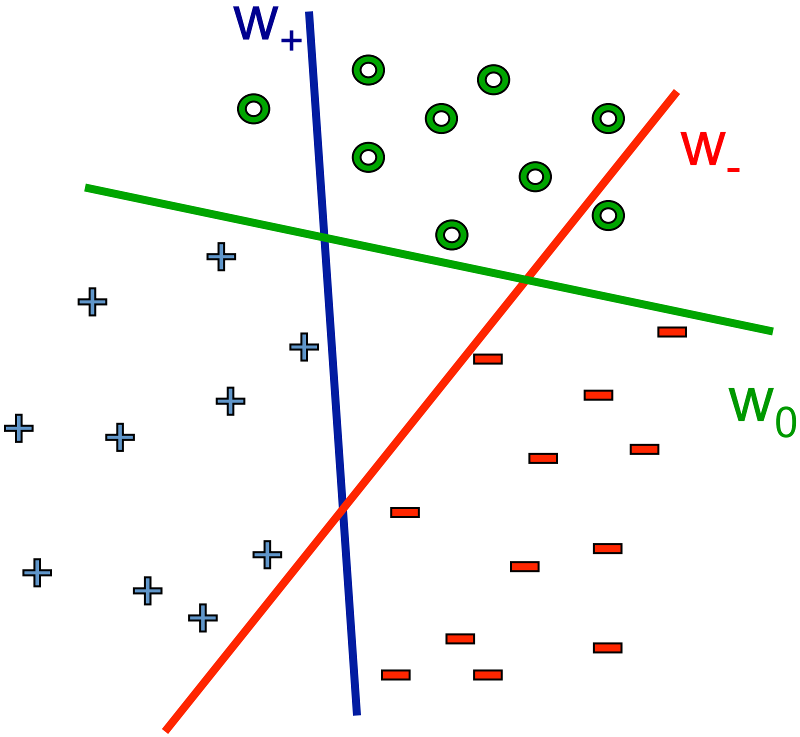
$$y^j f(x^j)$$

We want to smoothly approximate 0/1 loss!

What about multiple classes?



One against All



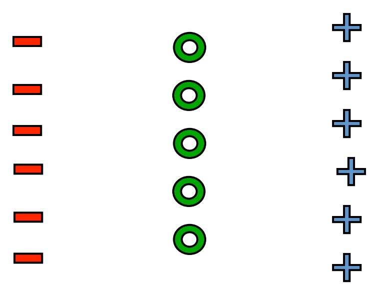
Learn 3 classifiers:

- \+ vs {0,-}, weights w_+
- \- vs {0,+}, weights w_-
- 0 vs {+,-}, weights w_0

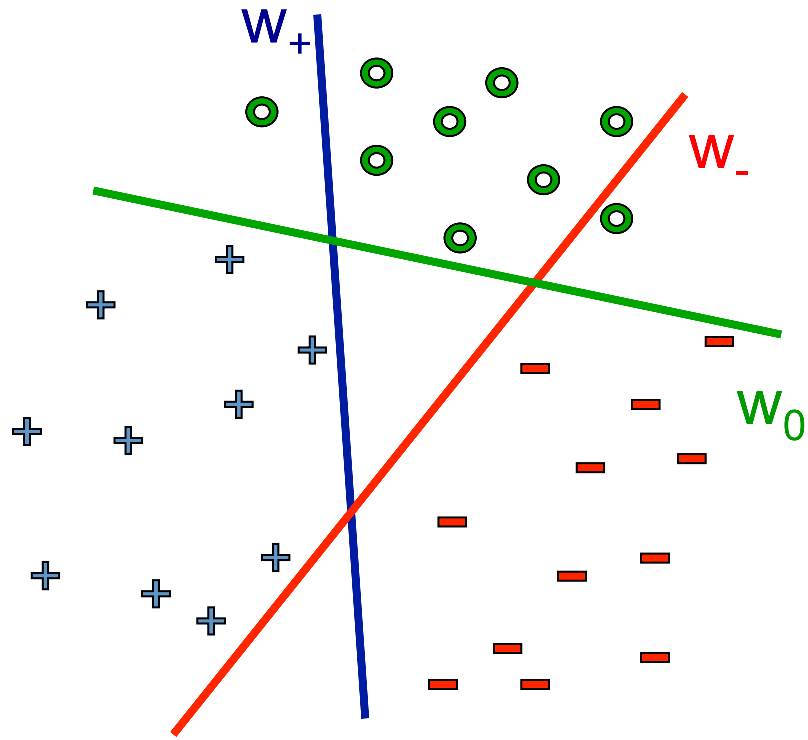
Output for x:

$y = \text{argmax}_i \ w_i \cdot x$

Any problems?
Could we learn this → dataset?



Learn 1 classifier: Multiclass SVM



Simultaneously learn 3 sets of weights:

- How do we guarantee the correct labels?
- Need new constraints!

For each class:

$$w^{y^j} \cdot x^j + w_0^{y^j} \geq w^{y'} \cdot x^j + w_0^{y'} + 1, \quad \forall y' \neq y^j, \quad \forall j$$

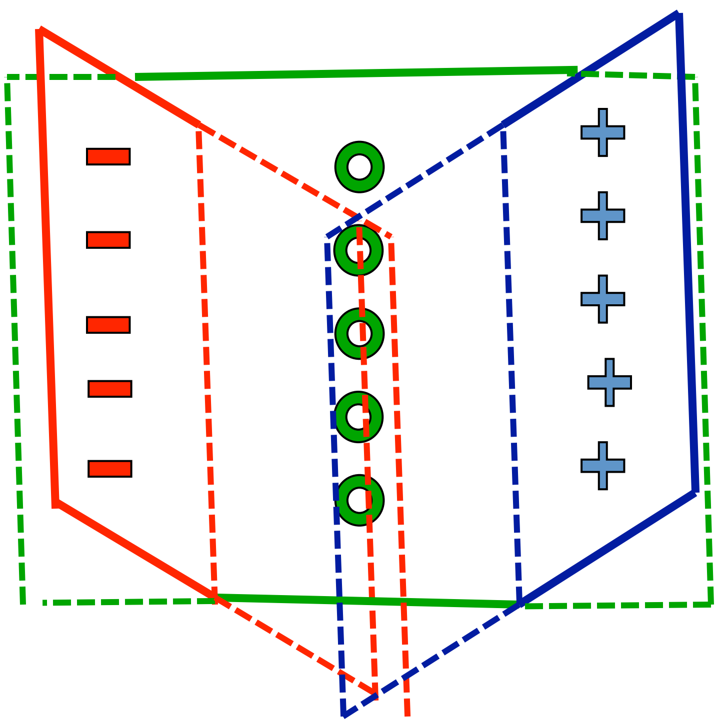
Learn 1 classifier: Multiclass SVM

Also, can introduce slack variables, as before:

$$\min_{w,w_0} \sum_y \|w^y\|_2^2 + C\sum_j \xi^j$$

$$w^{y^j} \cdot x^j + w_0^{y^j} \geq w^{y'} \cdot x^j + w_0^{y'} + 1 - \xi^j, \quad \forall y' \neq y^j, \quad \xi^j > 0 \;\; \forall j$$

Now, can we learn it? →



What you need to know

- Maximizing margin
- Derivation of SVM formulation
- Slack variables and hinge loss
- Tackling multiple class
  - One against All
  - Multiclass SVMs